SOLUTIONS OF SYSTEMS OF ELLIPTIC DIFFERENTIAL EQUATIONS ON CIRCULAR DOMAINS

JOANNA GAWRYCKA AND SŁAWOMIR RYBICKI†

Abstract. We study global bifurcation of weak solutions of systems of elliptic differential equations considered on $SO(2)-$invariant domains. We formulate sufficient conditions for the existence of unbounded continua of nontrivial solutions branching from the trivial ones. As the main tool we use the degree for $SO(2)-$equivariant gradient maps defined by the second author in [26].

1. Introduction

Rabinowitz global bifurcation theorem considers problems of the form

$$u = \lambda Lu + N(u, \lambda) \tag{1.1}$$

where $\lambda \in \mathbb{R}, u \in \mathbb{X}, \mathbb{X}$ is a real Banach space, $L : \mathbb{X} \to \mathbb{X}$ is a compact linear operator and $N : \mathbb{X} \times \mathbb{R} \to \mathbb{X}$ is a compact nonlinear operator with $N(u, \lambda) = o(\|u\|)$ for u near $0 \in \mathbb{E}$ uniformly on compact λ intervals. Let $\chi(L)$ denote the set of characteristic values of L. Solutions of (1.1) of the form $(0, \lambda) \in \mathbb{X} \times \mathbb{R}$ are called trivial. Define the set of nontrivial solutions of (1.1) as follows

$$\mathcal{N} = \{(u, \lambda) \in (\mathbb{X} \setminus \{0\}) \times \mathbb{R} : u = \lambda Lu + N(u, \lambda)\}.$$

Fix $\lambda_i \in \chi(L)$ and denote by $\mathcal{BIF}_{LS}(\lambda_i) \in \{0, \pm 2\} \subset \mathbb{Z}$ the bifurcation index at $(\lambda_i, 0)$ computed in terms of the Leray-Schauder topological degree. The classical Rabinowitz alternative can be formulated in the following way.

Theorem 1.1 ([20])**.** *If $\lambda_0 \in \chi(L)$ has odd algebraic multiplicity, then there is a maximal subcontinuum $\mathcal{C}_{\lambda_0} \subset closure(\mathcal{N})$ such that $(0, \lambda_0) \in \mathcal{C}_{\lambda_0}$ and either*

(1) *$\mathcal{C}_{\lambda_0}$ is unbounded, or*
(2) *$\mathcal{C}_{\lambda_0} \cap (\{0\} \times \mathbb{R}) = \{(0, \lambda_0), (0, \lambda_1), \ldots, (0, \lambda_k)\} \subset (\{0\} \times \chi(L)) \subset (\{0\} \times \mathbb{R})$; moreover*

$$\sum_{i=0}^{k} \mathcal{BIF}_{LS}(\lambda_i) = 0 \in \mathbb{Z} \tag{1.2}$$

In other words the Rabinowitz global bifurcation theorem shows that for a large class of nonlinear eigenvalue problems a continuum $\mathcal{C}$ (i.e. a closed, connected set) of solutions

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bifurcates from the trivial solution at each characteristic value (eigenvalue) of odd multiplicity of the problem linearized at the trivial solution. Each continuum (1) must either be unbounded, or (2) must meet some other characteristic value (eigenvalue). Moreover, the sum of bifurcation indices computed at characteristic values which belong to the bounded continuum $\mathcal{C}$ equals $0 \in \mathbb{Z}$. In other words a local linearized analysis forces the existence of a global bifurcation. This is a very powerful results that is quoted very often.

Many authors have answered the following question

How to exclude possibility (2)?

On one hand to exclude possibility (2) authors studied the nodal properties and symmetries of bifurcating solutions which are preserved on global continua of solutions.

It was shown in [23], using the nodal properties of solutions, that for the nonlinear Sturm-Liouville problem possibility (2) can not hold for the bifurcating continua in this problem. Moreover, it was shown in [23] that the continuum of positive solutions of elliptic differential equations emanating from the principal eigenvalue does not satisfy possibility (2). Many interesting results in this area have been proved, by using the nodal properties and symmetries of bifurcating solutions, by Healey and Kielhöfer in [11, 12, 13, 14, 17] and Rynne in [31].

On the other hand Cosner has considered in [5], under some additional assumptions, class of elliptic differential equations and has proved that the Morse index of solutions is invariant along continua of solutions of these equations. Which implies that possibility (2) can never occurs.

Summing up, all the above mentioned authors have studied properties of solutions of differential equations which were invariant along continua of solutions of these equations. Since different continua of solutions have possessed different properties, these continua have been separated and therefore unbounded.

Consider nonlinear eigenvalue problem (1.1) and assume additionally that $\mathbb{X}$ is a Hilbert space which is an orthogonal representation of the group $SO(2)$ and that this problem is $SO(2)-$invariant and possesses variational structure. We have proved the Rabinowitz global bifurcation theorem for the class of $SO(2)-$equivariant gradient (orthogonal) maps, see [26]. In this case the bifurcation index $\mathcal{BIF}(\lambda_i)$ is an element of the tom Dieck ring $U(SO(2))$. Since the ring structure in $U(SO(2))$ is much more complicated than the ring structure in $\mathbb{Z}$, it can happen that the sum of any finite number of bifurcation indices is nontrivial in $U(SO(2))$. In other words we can prove that all the bifurcating continua of solutions are unbounded. Notice that, it can happen that some of them are not separated.

We have used this idea in [27, 29, 30]. In article [27] we have studied global and symmetry-breaking solutions of elliptic differential equations on an annulus. We have proved the existence of unbounded continua of nontrivial solutions with symmetry-breaking phenomenon. In [29] we have considered equation of the form $-\Delta_{S^{n-1}} u = \lambda f(u)$, where $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on S^{n-1}, and have proved that any continuum of nontrivial solutions bifurcating from the trivial ones is unbounded in $\mathbb{H}^1(S^{n-1}) \times \mathbb{R}$. In [30] we have studied system of elliptic differential equations on $SO(2)-$invariant domain and have proved that if the number m of equations of this system is even then any continuum of nontrivial solutions emanating from the essential eigenvalue is unbounded in $\bigoplus_{i=1}^{m} \mathbb{H}_0^1(\Omega)$.

In this article we study global bifurcations of solutions of system of elliptic differential equations

$$\left\{ \begin{array}{rcll} -\Delta u & = & \nabla_u F(u, \lambda) & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right. \tag{1.3}$$

on an $SO(2)-$symmetric domain Ω.

We remark that there is a vast literature on the subject of reaction-diffusion systems, including the steady state situation as above, in view of its applications to chemical, biological and physical phenomena among others. Here we mention only [4, 18, 25].

Since we assume that considered system of equations possesses variational structure, as the main tool we use the degree for $SO(2)-$equivariant gradient maps and the Rabinowitz alternative for $SO(2)-$equivariant gradient maps, see [26]. We prove the sufficient conditions for the existence of unbounded continua of nontrivial solutions emanating from the trivial ones and the necessary conditions for the existence of bounded continua. In this article we generalize results of [29].

After introduction this article is organized as follows.

In Section 2 we have compiled basic facts on the degree for $SO(2)-$equivariant gradient maps.

Section 3 is devoted to the study of systems of linear equations (3.1), (3.2). In this section we perform local computations of bifurcation indices. In Lemma 3.6 we formulate the necessary condition for the existence of a bifurcation point. The formula for the bifurcation index, computed in terms of the degree for $SO(2)-$equivariant gradient maps, is proved in Lemma 3.7. The notion of an essential pair is introduced in Definition 3.1. Properties of the bifurcation index are described in Lemmas 3.9, 3.10.

Section 4 contains the main results of this article. In this section we study system of elliptic nonlinear differential equations (4.1). Theorem 4.1 is the Rabinowitz alternative for solutions of system (4.1). In Corollary 4.1 we formulate sufficient conditions for the existence of unbounded continua of nonzero solutions of system (4.1). Corollary 4.2 contains necessary conditions for the existence of bounded continua of nonzero solutions of system (4.1).

To illustrate the abstract results proved in this article in Section 5 we consider system (4.1) for $\Omega = B^2, B^3$.

In Section 6 we present some comments and final remarks.

2. $\nabla_{SO(2)}-$degree and Its Properties

In this section, for the convenience of the reader, we remind the main properties of the the degree for $SO(2)-$equivariant gradient maps. We denote it briefly by $\nabla_{SO(2)}-\deg$. Put $U(SO(2)) = \mathbb{Z} \oplus \left(\bigoplus_{k=1}^{\infty} \mathbb{Z} \right)$ and define actions $+, \star : U(SO(2)) \times U(SO(2)) \to U(SO(2))$ as follows

$$\alpha + \beta = (\alpha_0 + \beta_0, \alpha_1 + \beta_1, \dots, \alpha_k + \beta_k, \dots) \tag{2.1}$$
$$\alpha \star \beta = (\alpha_0 \cdot \beta_0, \alpha_0 \cdot \beta_1 + \beta_0 \cdot \alpha_1, \dots, \alpha_0 \cdot \beta_k + \beta_0 \cdot \alpha_k, \dots) \tag{2.2}$$

where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_k, \dots), \beta = (\beta_0, \beta_1, \dots, \beta_k, \dots) \in U(SO(2))$. It is easy to check that $(U(SO(2)), +, \star)$ is a commutative ring with unit. Ring $(U(SO(2)), +, \star)$ is known as the tom Dieck ring of the group $SO(2)$. For a definition of the tom Dieck ring $U(G)$ for any compact Lie group G we refer the reader to [8]. Additionally define $U_\pm(SO(2)) \subset U(SO(2))$ as follows

$$U_\pm(SO(2)) = \{\alpha \in U(SO(2)) : \pm\alpha_k \geq 0 \quad \text{for all} \quad k \in \mathbb{N} \cup \{0\}\}.$$

Remark 2.1. *It is easy to check that* $\star : U_\mp(SO(2)) \times U_-(SO(2)) \to U_\pm(SO(2))$.

In the following lemma we collect some basic properties of the tom Dieck ring $U(SO(2))$.

Lemma 2.1. *Let* $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_k, \dots), \beta = (\beta_0, \beta_1, \dots, \beta_k, \dots) \in U(SO(2))$. *Then,*

(1) $\mathbb{I} = (1, 0, \dots) \in U(SO(2))$ *is the unit in* $U(SO(2))$,
(2) $\alpha_0 = \pm 1$ *iff* α *is invertible in* $U(SO(2))$,
(3) *if* $\alpha_\pm = (\pm 1, \alpha_1, \dots, \alpha_k, \dots)$, *then* $\alpha_\pm^{-1} = (\pm 1, -\alpha_1, \dots, -\alpha_k, \dots)$,
(4) *if* $\alpha_\pm = (\pm 1, \alpha_1, \dots, \alpha_k, \dots)$, *then*
  (a) $\alpha_+^n = (1, n\alpha_1, \dots, n\alpha_k, \dots)$,
  (b) $\alpha_-^n = ((-1)^n, (-1)^{n+1}n\alpha_1, \dots, (-1)^{n+1}n\alpha_k, \dots)$.
(5) *if* $\alpha_0 = \beta_0 = 0$, *then* $\alpha \star \beta = \Theta \in U(SO(2))$.

An easy proof of this lemma is left to the reader.

If $\delta_1, \dots, \delta_q \in U(SO(2))$, then $\prod_{j=1}^{q} \delta_j = \delta_1 \star \dots \star \delta_q$. Moreover, it is understood that $\prod_{j \in \emptyset} \delta_j = \mathbb{I} \in U(SO(2))$.

Let V be a real, finite-dimensional, orthogonal representation of the group $SO(2)$. Define

- $C^k(V, \mathbb{R}) = \{f : V \to \mathbb{R} : f \text{ is a } C^k \text{ map}\}$,
- $C^k_{SO(2)}(V, \mathbb{R}) = \{f \in C^k(V, \mathbb{R}) : f \text{ is } SO(2)\text{-invariant}\}$,
- $GL_{SO(2)}(V) = \{L \in Aut(V) : L \text{ is } SO(2)\text{-equivariant}\}$,
- $GL^\nabla_{SO(2)}(V) = \{L \in GL_{SO(2)}(V) : \langle Lv, w\rangle = \langle v, Lw\rangle \text{ for any } v, w \in V\}$.

Let $f \in C^1_{SO(2)}(V, \mathbb{R})$. Since V is an orthogonal representation, $\nabla f : V \to V$ is an $SO(2)-$equivariant C^0-map. Choose an open, bounded and $SO(2)-$invariant subset $\Omega \subset V$ such that $(\nabla f)^{-1}(0) \cap \partial\Omega = \emptyset$. Under these assumptions we have defined in [26] the degree for $SO(2)-$equivariant gradient maps $\nabla_{SO(2)}-\deg(\nabla f, \Omega) \in U(SO(2))$ with coordinates

$$\nabla_{SO(2)}-\deg(\nabla f, \Omega) =$$
$$= (\nabla_{SO(2)}-\deg_{SO(2)}(\nabla f, \Omega), \nabla_{SO(2)}-\deg_{\mathbb{Z}_1}(\nabla f, \Omega), \dots, \nabla_{SO(2)}-\deg_{\mathbb{Z}_k}(\nabla f, \Omega), \dots).$$

Throughout this article $\gamma > 0$ and $D_\gamma(V) = \{v \in V : | v | < \gamma\}$. In the following theorem we formulate the main properties of the degree of $SO(2)-$equivariant gradient maps.

Theorem 2.1 ([26]). *Under the above assumptions the degree for $SO(2)-$equivariant gradient maps has the following properties:*

(1) *if* $\nabla_{SO(2)}-\deg(\nabla f, \Omega) \neq \Theta$, *then* $(\nabla f)^{-1}(0) \cap \Omega \neq \emptyset$,
(2) *if* $\nabla_{SO(2)}-\deg_H(\nabla f, \Omega) \neq 0$, *then* $(\nabla f)^{-1}(0) \cap \Omega^H \neq \emptyset$, *where* Ω^H *denotes the set of fixed points of the action of the subgroup* $H \subset SO(2)$ *on* Ω,
(3) *if* $\Omega = \Omega_0 \cup \Omega_1$ *and* $\Omega_0 \cap \Omega_1 = \emptyset$, *then*
$$\nabla_{SO(2)}-\deg(\nabla f, \Omega) = \nabla_{SO(2)}-\deg(\nabla f, \Omega_0) + \nabla_{SO(2)}-\deg(\nabla f, \Omega_1),$$
(4) *if* $\Omega_0 \subset \Omega$ *is an open* $SO(2)-$*equivariant subset and* $(\nabla f)^{-1}(0) \cap \Omega \subset \Omega_0$, *then*
$$DEG(\nabla f, \Omega) = DEG(\nabla f, \Omega_0),$$
(5) *if* $f \in C^1_{SO(2)}(V \times [0,1], \mathbb{R})$ *is such that* $(\nabla_v f)^{-1}(0) \cap (\partial\Omega \times [0,1]) = \emptyset$, *then*
$$\nabla_{SO(2)}-\deg(\nabla f_0, \Omega) = \nabla_{SO(2)}-\deg(\nabla f_1, \Omega),$$
(6) *if* W *is an orthogonal representation of the group* $SO(2)$, *then*
$$\nabla_{SO(2)}-\deg((\nabla f, Id), \Omega \times D_\gamma(W)) = \nabla_{SO(2)}-\deg(\nabla f, \Omega),$$
(7) *if* $f \in C^2_{SO(2)}(V, \mathbb{R})$ *is such that* $\nabla f(0) = 0$ *and* $\nabla^2 f(0)$ *is an* $SO(2)-$*equivariant self-adjoint isomorphism then there is* $\gamma > 0$ *such that*
$$\nabla_{SO(2)}-\deg(\nabla f, D_\gamma(V)) = \nabla_{SO(2)}-\deg(\nabla^2 f(0), D_\gamma(V)).$$

Below we formulate product formula for the degree for $SO(2)-$equivariant gradient maps.

Theorem 2.2 ([28]). *Let* $\Omega_i \subset V_i$ *be an open, bounded and* $SO(2)-$*invariant subset of* $SO(2)-$*representation* $V_i, i = 1, 2$. *Let* $f_i \in C^1_{SO(2)}(V_i, \mathbb{R})$ *be such that* $(\nabla f_i)^{-1}(0) \cap \partial\Omega_i = \emptyset, i = 1, 2$. *Then*
$$\nabla_{SO(2)}-\deg((\nabla f_1, \nabla f_2), \Omega_1 \times \Omega_2) = \nabla_{SO(2)}-\deg(\nabla f_1, \Omega_1) \star \nabla_{SO(2)}-\deg(\nabla f_2, \Omega_2).$$

For $j \in \mathbb{N}$ define a map $\rho^j : SO(2) \to GL(2, \mathbb{R})$ as follows
$$\rho^j(e^{i\cdot\theta}) = \begin{bmatrix} \cos j\cdot\theta & -\sin j\cdot\theta \\ \sin j\cdot\theta & \cos j\cdot\theta \end{bmatrix} \qquad 0 \leq \theta < 2\cdot\pi.$$

For $k,\ j \in \mathbb{N}$ we denote by $\mathbb{R}[k, j]$ the direct sum of k copies of $(\mathbb{R}^2, \rho^j)$, we also denote by $\mathbb{R}[k, 0]$ the trivial $k-$dimensional representation of $SO(2)$. We say that two representations V and W are equivalent if there exists an equivariant, linear isomorphism $T : V \to W$. The following classic result gives a complete classification (up to equivalence) of finite–dimensional representations of the group $SO(2)$ (see [1]).

Theorem 2.3 ([1]). *If* V *is a finite-dimensional representation of the group* $SO(2)$ *then there exist finite sequences* $\{k_i\}$, $\{j_i\}$ *satisfying* $(*)$ $\quad j_i \in \{0\} \cup \mathbb{N}, \quad k_i \in \mathbb{N}, \quad 1 \leq i \leq r$, $j_1 < j_2 < \cdots < j_r$ *such that the representation* V *is equivalent to the representation* $\bigoplus_{i=1}^{r} \mathbb{R}[k_i, j_i]$. *Moreover, the equivalence class of* V ($V \approx \bigoplus_{i=1}^{r} \mathbb{R}[k_i, j_i]$) *is uniquely determined by* $\{j_i\}$, $\{k_i\}$ *satisfying* $(*)$.

We will denote by $m^-(L)$ the Morse index of a symmetric matrix L.
To apply successfully any topological degree we need computational formulas for this invariant. Below we show how to compute the degree for $SO(2)-$equivariant gradient maps of linear, self-adjoint, $SO(2)-$equivariant isomorphism.

Lemma 2.2 ([26]). *If $V \approx \mathbb{R}[k_0,0] \oplus \mathbb{R}[k_1,m_1] \oplus \ldots \oplus \mathbb{R}[k_r,m_r]$, $L \in GL^{\nabla}_{SO(2)}(V)$ and $\gamma > 0$ then*

(1) $L = \text{diag}\,(L_0, L_1, \ldots, L_r)$,
(2)

$$\nabla_{SO(2)}\text{-}\deg_H(L, D_\gamma(V)) = \begin{cases} (-1)^{m^-(L_0)}, & \text{for } H = SO(2), \\ (-1)^{m^-(L_0)} \cdot \dfrac{m^-(L_i)}{2}, & \text{for } H = \mathbb{Z}_{m_i}, \\ 0, & \text{for } H \notin \{SO(2), \mathbb{Z}_{m_1}, \ldots, \mathbb{Z}_{m_r}\}, \end{cases}$$

(3) in particular, if $L = -Id$, then

$$\nabla_{SO(2)}\text{-}\deg_H(-Id, D_\gamma(V)) = \begin{cases} (-1)^{k_0}, & \text{for } H = SO(2), \\ (-1)^{k_0} \cdot k_i, & \text{for } H = \mathbb{Z}_{m_i}, \\ 0, & \text{for } H \notin \{SO(2), \mathbb{Z}_{m_1}, \ldots, \mathbb{Z}_{m_r}\}. \end{cases}$$

The following lemma is a direct consequence of Lemmas 2.1, 2.2.

Lemma 2.3. *If $V \approx \mathbb{R}[k_0,0] \oplus \mathbb{R}[k_1,m_1] \oplus \ldots \oplus \mathbb{R}[k_r,m_r], L \in GL^{\nabla}_{SO(2)}(V)$ and $\gamma > 0$ then*

(1) $\nabla_{SO(2)}\text{-}\deg(L, D_\gamma(V))$ *is invertible in* $U(SO(2))$,
(2) $(-1)^{m^-(L_0)} \cdot \nabla_{SO(2)}\text{-}\deg(L, D_\gamma(V)) \in U_+(SO(2))$,
(3) $\left(\nabla_{SO(2)}\text{-}\deg(L, D_\gamma(V))\right)^{2n} \in U_+(SO(2))$ *for any* $n \in \mathbb{N}$,
(4) $(-1)^{k_0} \cdot \nabla_{SO(2)}\text{-}\deg(-Id, D_\gamma(V)) \in U_+(SO(2))$,
(5) *if* $n \in \mathbb{N}$, *then*

$$\left(\left(\nabla_{SO(2)}\text{-}\deg(-Id, D_\gamma(V))\right)^{2n}\right)_H = \begin{cases} 1, & \text{for } H = SO(2), \\ 2 \cdot n \cdot k_i, & \text{for } H = \mathbb{Z}_{m_i}, \\ 0, & \text{for } H \notin \{SO(2), \mathbb{Z}_{m_1}, \ldots, \mathbb{Z}_{m_r}\}. \end{cases}$$

Let $(\mathbb{H}, \langle \cdot, \cdot \rangle_{\mathbb{H}})$ be an infinite-dimensional, separable Hilbert space which is an orthogonal representation of the group $SO(2)$ and let $C^1_{SO(2)}(\mathbb{H}, \mathbb{R})$ denote the set of $SO(2)-$invariant C^1-functionals. Fix $\Phi \in C^1_{SO(2)}(\mathbb{H}, \mathbb{R})$ such that

$$\nabla\Phi(u) = u - \nabla\eta(u), \tag{2.3}$$

where $\nabla\eta : \mathbb{H} \to \mathbb{H}$ is an $SO(2)-$equivariant compact operator. Let $\Omega \subset \mathbb{H}$ be an open bounded and $SO(2)-$invariant set such that $(\nabla\Phi)^{-1}(0) \cap \partial\Omega = \emptyset$. In this situation $\nabla_{SO(2)}\text{-}\deg(Id - \nabla\eta, \Omega) \in U(SO(2))$ is well-defined, see [26].
Let $L : \mathbb{H} \to \mathbb{H}$ be a linear, bounded, self-adjoint, $SO(2)-$equivariant operator with spectrum $\sigma(L) = \{\lambda_i\}$. By $V_L(\lambda_i)$ we will denote eigenspace of L corresponding to eigenvalue λ_i and we put $\mu(\lambda_i) = \dim V_L(\lambda_i)$. In other words $\mu(\lambda_i)$ is the multiplicity of λ_i. Since

operator L is linear, bounded, self-adjoint, and $SO(2)-$equivariant, $V_L(\lambda_i)$ is a finite-dimensional representation of the group $SO(2)$. Define $D_\gamma(\mathbb{H}) = \{h \in \mathbb{H} : \|h\|_{\mathbb{H}} < \gamma\}$. Combining Theorem 4.5 in [26] with Theorem 2.2 we obtain the following theorem.

Theorem 2.4. *Under the above assumptions if* $1 \notin \sigma(L)$, *then*

$$\nabla_{SO(2)}-\deg(Id - L, D_\gamma(\mathbb{H})) = \prod_{\lambda_i > 1} \nabla_{SO(2)}-\deg(-Id, D_\gamma(V_L(\lambda_i))).$$

By $C^1_{SO(2)}(\mathbb{H}\times\mathbb{R},\mathbb{R})$ we will denote the set of families of $SO(2)-$invariant C^1-functionals. Let functional $\Phi \in C^1_{SO(2)}(\mathbb{H}\times\mathbb{R},\mathbb{R})$ be such that $\Phi(u,\lambda) = \frac{1}{2}\langle u - \lambda L u, u\rangle_{\mathbb{H}} + \eta(u,\lambda)$, where $L : \mathbb{H} \to \mathbb{H}$ is an $SO(2)-$equivariant, linear, self-adjoint, compact operator and $\nabla_u \eta : \mathbb{H}\times\mathbb{R} \to \mathbb{H}$ is an $SO(2)-$equivariant compact operator and such that

a) $\nabla_u\eta(0,\lambda) = 0$, for all $\lambda \in \mathbb{R}$,
b) $\nabla_u\eta(h,\lambda) = o(\|h\|)$, uniformly on bounded $\lambda-$intervals.

Put $\mathcal{N}(\Phi) = \{(u,\lambda) \in (\mathbb{H}\setminus\{0\})\times\mathbb{R} : \nabla_u\Phi(u,\lambda) = 0\}$. Let $\mathcal{C}(\lambda_0)$ denote connected component of closure$(\mathcal{N}(\Phi))$ such that $(0,\lambda_0) \in \mathcal{C}(\lambda_0)$.

Definition 2.1. *A point* $(0,\lambda_0) \in \mathbb{H}\times\mathbb{R}$ *is said to be a branching point of solutions of the equation* $\nabla_u\Phi(u,\lambda) = 0$, *if* $\mathcal{C}(\lambda_0)\setminus\{(0,\lambda_0)\} \neq \emptyset$. *A point* $(0,\lambda_0) \in \mathbb{H}\times\mathbb{R}$ *is said to be a bifurcation point of solutions of the equation* $\nabla_u\Phi(u,\lambda) = 0$, *if* $(0,\lambda_0) \in$ closure$(\mathcal{N}(\Phi))$.

Of course any branching point is a bifurcation point. It is worth to point out that there are bifurcation points which are not branching points.

Remark 2.2. *Suppose that* $\Phi \in C^2_{SO(2)}(\mathbb{H},\mathbb{R})$. *Is is well-known that if* $(0,\lambda_0) \in \mathbb{H}\times\mathbb{R}$ *is a bifurcation point of solutions of the equation* $\nabla_u\Phi(u,\lambda) = 0$, *then* $\nabla_u^2\Phi(0,\lambda_0)$ *is not an isomorphism. In other words if* $(0,\lambda_0) \in \mathbb{H}\times\mathbb{R}$ *is a bifurcation point of solutions of the equation* $\nabla_u\Phi(u,\lambda) = 0$, *then* λ_0 *is a characteristic value of* L.

Fix $\lambda_{i_0} \in \sigma(L)$. Choose $\epsilon > 0$ such that $\lambda_{i_0}^{-1}$ is the only characteristic value of L in $[\lambda_{i_0}^{-1} - \varepsilon, \lambda_{i_0}^{-1} + \varepsilon]$ and define a bifurcation index $\mathcal{BIF}\left(\lambda_{i_0}^{-1}\right) \in U(SO(2))$ as follows

$$\mathcal{BIF}\left(\lambda_{i_0}^{-1}\right) =$$

$$= \nabla_{SO(2)}-\deg(Id - \left(\lambda_{i_0}^{-1} + \varepsilon\right) L, D_\gamma(\mathbb{H})) - \nabla_{SO(2)}-\deg(Id - \left(\lambda_{i_0}^{-1} - \varepsilon\right) L, D_\gamma(\mathbb{H})).$$

Denote

(1) $\sigma_+(L) = \sigma(L) \cap (0,+\infty) = \{\lambda_1^+, \lambda_2^+, \ldots, \lambda_i^+, \ldots\}$,
(2) $\sigma_-(L) = \sigma(L) \cap (-\infty,0) = \{\lambda_1^-, \lambda_2^-, \ldots, \lambda_i^-, \ldots\}$,

with order $\lambda_1^- < \lambda_2^- < \ldots < \lambda_i^- < \ldots < 0 < \ldots < \lambda_i^+ < \ldots < \lambda_2^+ < \lambda_1^+$.

Lemma 2.4. *Under the above assumptions the following formulas hold true:*

(1) $\mathcal{BIF}\left(\left(\lambda_1^+\right)^{-1}\right) = \nabla_{SO(2)}-\deg\left(-Id, D_\gamma\left(V_L\left(\lambda_1^+\right)\right)\right) - \mathbb{I}$,
(2) *for* $i_0 \geq 2$,

$$\mathcal{BIF}\left(\left(\lambda_{i_0}^+\right)^{-1}\right) =$$

$$= \nabla_{SO(2)}\text{-deg}\left(-Id, D_\gamma\left(\bigoplus_{i=1}^{i_0-1} V_L\left(\lambda_i^+\right)\right)\right) \star \left(\nabla_{SO(2)}\text{-deg}\left(-Id, D_\gamma\left(V_L\left(\lambda_{i_0}^+\right)\right)\right) - \mathbb{I}\right),$$

(3) $\mathcal{BIF}\left(\left(\lambda_1^-\right)^{-1}\right) = \mathbb{I} - \nabla_{SO(2)}\text{-deg}\left(-Id, D_\gamma\left(V_L\left(\lambda_1^-\right)\right)\right)$,

(4) *for* $i_0 \geq 2$,

$$\mathcal{BIF}\left(\left(\lambda_{i_0}^-\right)^{-1}\right) =$$

$$= \nabla_{SO(2)}\text{-deg}\left(-Id, D_\gamma\left(\bigoplus_{i=1}^{i_0-1} V_L\left(\lambda_i^-\right)\right)\right) \star \left(\mathbb{I} - \nabla_{SO(2)}\text{-deg}\left(-Id, D_\gamma\left(V_L\left(\lambda_{i_0}^-\right)\right)\right)\right).$$

The proof of the above lemma is in fact direct consequence of Theorem 2.4.
The following theorem is analogous to the classical Rabinowitz global bifurcation theorem. Rabinowitz alternative has been proved, by using the Leray Schauder degree, for the operators in the form compact perturbation of the identity. In our theorem we have assumed additionally that operators are potential and $SO(2)-$equivariant. Therefore to prove this theorem we have used infinite dimensional version of the the degree for $SO(2)-$equivariant gradient maps. In other words we study global bifurcations of critical points of $SO(2)-$invariant functionals. We formulate sufficient conditions for the existence of branching points of critical points of functionals. Moreover, we study global properties of closed connected sets of critical points.

Theorem 2.5 ([26])**.** *Fix* $\lambda_{i_0} \in \sigma(L) \setminus \{0\}$ *such that* $\mathcal{BIF}\left(\lambda_{i_0}^{-1}\right) \neq \Theta \in U(SO(2))$. *Then*

(1) *either* $\mathcal{C}\left(\lambda_{i_0}^{-1}\right)$ *is unbounded in* $\mathbb{H} \times \mathbb{R}$,

(2) *or* $\mathcal{C}\left(\lambda_{i_0}^{-1}\right)$ *is bounded in* $\mathbb{H} \times \mathbb{R}$ *and additionally the following conditions are satisfied*

(a) $\mathcal{C}\left(\lambda_{i_0}^{-1}\right) \cap (\{0\} \times \mathbb{R}) = \{0\} \times \left(\bigcup_{j=0}^{p} \left\{\lambda_{i_j}^{-1}\right\}\right)$,

(b) $\sum_{j=0}^{p} \mathcal{BIF}\left(\lambda_{i_j}^{-1}\right) = \Theta \in U(SO(2))$.

3. Linear equation

One of the most important theorems in topological nonlinear analysis is the Rabinowitz global bifurcation theorem. Rabinowitz discovered that under certain circumstances a local linearized analysis forces the existence of branching point of nontrivial solutions of nonlinear eigenvalue problem. This is a very powerful result that is quoted very often. More precisely, Rabinowitz has proved that nontriviality of the bifurcation index implies the existence of a closed connected set of nontrivial solutions, branching from the set of trivial solutions, which is either unbounded or come back to the set of trivial solutions.

In this section we consider system of linear elliptic differential equations. The aim of this section is to study linear operator induced by this system and to compute bifurcation indices given by Lemma 2.4.

Let A be a symmetric $(m \times m)-$matrix and $\Omega \subset \mathbb{R}^N$ be an open and bounded subset. Consider the following system of linear elliptic differential equations

$$\begin{cases} -\Delta u = & Au \quad \text{in } \Omega \\ \quad u = & 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.1}$$

Define a separable Hilbert space $\mathbb{H} = \bigoplus_{i=1}^{m} \mathbb{H}_0^1(\Omega)$ with scalar product

$$\langle u, v\rangle_{\mathbb{H}} = \sum_{i=1}^{m} \langle u_i, v_i\rangle_{\mathbb{H}_0^1(\Omega)} = \int_\Omega \nabla u(x)\nabla v(x)\, dx,$$

for $u = (u_1, \ldots, u_m), v = (v_1, \ldots, v_m) \in \mathbb{H}$.

Remark 3.1. *If we consider $\mathbb{R}^N$ as an orthogonal representation of the group $SO(2)$ and $\Omega \subset \mathbb{R}^N$ is an open, bounded and $SO(2)-$invariant subset then $(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ is an orthogonal representation of the group $SO(2)$ with an action given by $g \cdot u(x) = u(gx)$.*

Define functional $\Phi \in C^1_{SO(2)}(\mathbb{H}, \mathbb{R})$ as follows $\Phi(u) = \frac{1}{2}\langle u, u\rangle_{\mathbb{H}} - \frac{1}{2}\int_\Omega Au(x) \cdot u(x)\, dx$.

Lemma 3.1. *The gradient $\nabla\Phi : \mathbb{H} \to \mathbb{H}$ is a linear, self-adjoint, $SO(2)-$equivariant operator of the form $\nabla\Phi(u) = u - L_A u$, where $L_A = A \cdot ((-\Delta)^{-1} \cdot Id_{\mathbb{R}^m}) : \mathbb{H} \to \mathbb{H}$ is a compact operator given by formula $\langle L_A u, v\rangle_{\mathbb{H}} = \int_\Omega Au(x)v(x)dx$.*

Let us denote by $\sigma(A)$ the spectrum of A. We will denote by $\mu(\alpha)$ the multiplicity of the eigenvalue $\alpha \in \sigma(A)$. Additionally, define $\sigma_+(A) = \sigma(A) \cap (0, +\infty)$ and $\sigma_-(A) = \sigma(A) \cap (-\infty, 0)$.
Denote by $\sigma(-\Delta; \Omega) = \{\lambda_k : 0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k < \ldots\}$ the eigenvalues of the following linear problem

$$\begin{cases} -\Delta u = & \lambda u \quad \text{in } \Omega, \\ \quad u = & 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Let $V_{-\Delta}(\lambda_k)$ denote eigenspace of $-\Delta$ corresponding to eigenvalue $\lambda_k \in \sigma(-\Delta; \Omega)$. Additionally put $\mu(\lambda_k) = \dim V_{-\Delta}(\lambda_k)$ i.e. $\mu(\lambda_k)$ is the multiplicity of λ_k.

Lemma 3.2. *The following conditions are equivalent:*

1. *system* (3.1) *possesses nonzero solution,*
2. *operator $Id - L_A : \mathbb{H} \to \mathbb{H}$ is not an isomorphism,*
3. $\sigma(-\Delta; \Omega) \cap \sigma_+(A) \neq \emptyset$.

The above lemma has been proved in [6] for $n = 2$. The same proof remains valid for $n > 2$.
The Jordan form of A will be denoted by $J(A)$. Denote by $S(m, \mathbb{R})$ the set of symmetric real $(m \times m)-$matrices.

Lemma 3.3. *Let $A \in S(m, \mathbb{R})$. Then there is a continuous path $A : [0, 1] \to S(m, \mathbb{R})$ such that $A(0) = A, A(1) = J(A)$ and $\sigma(A(t)) = \sigma(A)$ for any $t \in [0, 1]$.*

The easy proof of the above lemma is left to the reader.

Lemma 3.4. *Assume that* $\sigma(-\Delta;\Omega)\cap\sigma(A)=\emptyset$. *Then*

$$\nabla_{SO(2)}-\deg(Id-L_A,D_\gamma(\mathbb{H}))=\nabla_{SO(2)}-\deg(Id-L_{J(A)},D_\gamma(\mathbb{H}))=$$
$$=\prod_{\alpha_j\in\sigma(A)\cap(\lambda_1,+\infty)}\left(\nabla_{SO(2)}-\deg\left(Id-L_{\alpha_j},D_\gamma\left(\mathbb{H}^1_0(\Omega)\right)\right)\right)^{\mu(\alpha_j)},$$

where $L_{\alpha_j}=\alpha_j\cdot(-\Delta)^{-1}$.

Proof. By Lemmas 3.1, 3.2 the operator $Id-L_A:\mathbb{H}\to\mathbb{H}$ is an $SO(2)$−equivariant, self-adjoint isomorphism of the form compact perturbation of the identity. Let $A(t),t\in[0,1]$ be a path of symmetric matrices given by Lemma 3.3. Since $\sigma(A(t))=\sigma(A)$ for any $t\in[0,1]$, $Id-L_{A(t)}:\mathbb{H}\to\mathbb{H}$ is a family of $SO(2)$−equivariant, self-adjoint isomorphism of the form compact perturbation of the identity. Applying homotopy invariance of the degree for $SO(2)$−equivariant gradient maps we obtain $\nabla_{SO(2)}-\deg(Id-L_A,D_\gamma(\mathbb{H}))=\nabla_{SO(2)}-\deg(Id-L_{J(A)},D_\gamma(\mathbb{H}))$.

Since $Id-L_{J(A)}:\mathbb{H}=\bigoplus_{i=1}^{m}\mathbb{H}^1_0(\Omega)\to\mathbb{H}$ is a product map, we obtain

$$\nabla_{SO(2)}-\deg(Id-L_{J(A)},D_\gamma(\mathbb{H}))=\prod_{\alpha_j\in\sigma(A)}\left(\nabla_{SO(2)}-\deg\left(Id-L_{\alpha_j},D_\gamma\left(\mathbb{H}^1_0(\Omega)\right)\right)\right)^{\mu(\alpha_j)}=$$
$$=\prod_{\alpha_j\in\sigma(A)\cap(\lambda_1,+\infty)}\left(\nabla_{SO(2)}-\deg\left(Id-L_{\alpha_j},D_\gamma\left(\mathbb{H}^1_0(\Omega)\right)\right)\right)^{\mu(\alpha_j)}.$$

which completes the proof. □

For $\alpha\in\sigma(A)\cap(\lambda_1,+\infty)$ define $Q(\alpha)=\bigoplus_{\lambda_k<\alpha}V_{-\Delta}(\lambda_k)$.

Lemma 3.5. *Assume that* $\sigma(-\Delta;\Omega)\cap\sigma(A)=\emptyset$. *Then,*

$$\nabla_{SO(2)}-\deg(Id-L_A,D_\gamma(\mathbb{H}))=\prod_{\alpha_j\in\sigma(A)\cap(\lambda_1,+\infty)}\left(\nabla_{SO(2)}-\deg(-Id,D_\gamma(\bigoplus_{\lambda_k<\alpha_j}V_{-\Delta}(\lambda_k))\right)^{\mu(\alpha_j)}.$$

Proof. By Lemma 3.4 we have

$$\nabla_{SO(2)}-\deg(Id-L_A,D_\gamma(\mathbb{H}))=$$
$$=\prod_{\alpha_j\in\sigma(A)\cap(\lambda_1,+\infty)}\left(\nabla_{SO(2)}-\deg\left(Id-L_{\alpha_j},D_\gamma\left(\mathbb{H}^1_0(\Omega)\right)\right)\right)^{\mu(\alpha_j)}.$$

That is why applying Theorem 2.4 we obtain the following

$$\prod_{\alpha_j\in\sigma(A)\cap(\lambda_1,+\infty)}\left(\nabla_{SO(2)}-\deg\left(Id-L_{\alpha_j},D_\gamma\left(\mathbb{H}^1_0(\Omega)\right)\right)\right)^{\mu(\alpha_j)}=$$

$$= \prod_{\alpha_j \in \sigma(A)\cap(\lambda_1,+\infty)} \left(\nabla_{SO(2)} - \deg(-Id, D_\gamma(\bigoplus_{\lambda_k<\alpha_j} V_{-\Delta}(\lambda_k))\right)^{\mu(\alpha_j)},$$

which completes the proof. □

Let us consider family of linear elliptic differential equations of the form

$$\begin{cases} -\Delta u = & \lambda A u \quad \text{in } \Omega, \\ \quad u = & 0 \quad\quad \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Lemma 3.6. *The following conditions are equivalent*

(1) *equation* (3.3) *possesses nonzero solution,*

(2) $\lambda \in \bigcup_{\lambda_k \in \sigma(-\Delta;\Omega)} \bigcup_{\alpha \in \sigma(A)\setminus\{0\}} \left\{\frac{\lambda_k}{\alpha}\right\}$.

The proof of the above lemma is a direct consequence of a Lemma 3.2.
Fix $\frac{\lambda_{k_0}}{\alpha_{j_0}} \in \bigcup_{\lambda_k \in \sigma(-\Delta;\Omega)} \bigcup_{\alpha_j \in \sigma(A)\setminus\{0\}} \left\{\frac{\lambda_k}{\alpha_j}\right\}$. Notice that there is $\epsilon > 0$ such that

$$\left[\frac{\lambda_{k_0}}{\alpha_{j_0}} - \epsilon, \frac{\lambda_{k_0}}{\alpha_{j_0}} + \epsilon\right] \cap \left(\bigcup_{\lambda_k \in \sigma(-\Delta;\Omega)} \bigcup_{\alpha_j \in \sigma(A)\setminus\{0\}} \left\{\frac{\lambda_k}{\alpha_j}\right\} \right) = \left\{\frac{\lambda_{k_0}}{\alpha_{j_0}}\right\} \tag{3.4}$$

For $\alpha_j \in \sigma(A) \setminus \{0\}$ define $\mathcal{W}\left(\alpha_j, \frac{\lambda_{k_0}}{\alpha_{j_0}}\right) = \mathbb{R}[2,0] \oplus \bigoplus_{\lambda_k < \frac{\lambda_{k_0}}{\alpha_{j_0}}\alpha_j} V_{-\Delta}(\lambda_k)$.

In the following lemma we deliver formula for bifurcation index.

Lemma 3.7. *Fix* $\lambda_{k_0} \in \sigma(-\Delta;\Omega)$ *and* $\alpha_{j_0} \in \sigma_\pm(A)$. *Then*

$$\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) =$$

$$= \pm \prod_{\alpha_j \in \sigma_\pm(A)} \left(\nabla_{SO(2)} - \deg\left(-Id, D_\beta\left(\mathcal{W}\left(\alpha_j, \frac{\lambda_{k_0}}{\alpha_{j_0}}\right)\right)\right)\right)^{\mu(\alpha_j)} \star$$

$$\star \left(\left(\nabla_{SO(2)} - \deg(-Id, D_\beta(V_{-\Delta}(\lambda_{k_0})))\right)^{\mu(\alpha_{j_0})} - \mathbb{I}\right).$$

Proof. Suppose that $\alpha_{j_0} \in \sigma_+(A)$. Notice that from Lemma 3.4 and (3.4) it follows that for sufficiently small $\epsilon > 0$ we have

$$\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) =$$

$$= \nabla_{SO(2)} - \deg\left(Id - \left(\frac{\lambda_{k_0}}{\alpha_{j_0}} + \epsilon\right) L_A, D_\beta(\mathbb{H})\right) -$$

$$-\nabla_{SO(2)}-\deg\left(Id-\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}-\epsilon\right)L_A, D_\beta(\mathbb{H})\right)=$$
$$=\nabla_{SO(2)}-\deg\left(Id-\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}+\epsilon\right)L_{J(A)}, D_\beta(\mathbb{H})\right)-$$
$$-\nabla_{SO(2)}-\deg\left(Id-\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}-\epsilon\right)L_{J(A)}, D_\beta(\mathbb{H})\right)=$$
$$=\prod_{\alpha_j\in\sigma_+(A)}\left(\nabla_{SO(2)}-\deg\left(Id-\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}+\epsilon\right)L_{\alpha_j}, D_\beta\left(\mathbb{H}^1_0(\Omega)\right)\right)\right)^{\mu(\alpha_j)}-$$
$$-\prod_{\alpha_j\in\sigma_+(A)}\left(\nabla_{SO(2)}-\deg\left(Id-\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}-\epsilon\right)L_{\alpha_j}, D_\beta\left(\mathbb{H}^1_0(\Omega)\right)\right)\right)^{\mu(\alpha_j)}.$$

Consequently taking into account the above and Theorem 2.4 we obtain the following

$$\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)=$$
$$=\prod_{\alpha_j\in\sigma_+(A)\setminus\{\alpha_{j_0}\}}\left(\nabla_{SO(2)}-\deg\left(Id-\frac{\lambda_{k_0}}{\alpha_{j_0}}L_{\alpha_j}, D_\beta\left(\mathbb{H}^1_0(\Omega)\right)\right)\right)^{\mu(\alpha_j)}\star$$
$$\star\left(\left(\nabla_{SO(2)}-\deg\left(Id-\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}+\epsilon\right)L_{\alpha_{j_0}}, D_\beta\left(\mathbb{H}^1_0(\Omega)\right)\right)\right)^{\mu(\alpha_{j_0})}-\right.$$
$$\left.-\left(\nabla_{SO(2)}-\deg\left(Id-\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}-\epsilon\right)L_{\alpha_{j_0}}, D_\beta\left(\mathbb{H}^1_0(\Omega)\right)\right)\right)^{\mu(\alpha_{j_0})}\right)=$$
$$=\prod_{\alpha_j\in\sigma_+(A)}\left(\nabla_{SO(2)}-\deg\left(-Id, D_\beta\left(\mathcal{W}\left(\alpha_j,\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)\right)\right)\right)^{\mu(\alpha_j)}\star$$
$$\star\left(\left(\nabla_{SO(2)}-\deg\left(-Id, D_\beta(V_{-\Delta}(\lambda_{k_0}))\right)\right)^{\mu(\alpha_{j_0})}-\mathbb{I}\right),$$

which completes the proof.
Suppose that $\alpha_{j_0}\in\sigma_-(A)$. The proof of this case is in fact literally the same as proof presented above.

□

Let us formulate important consequences of Lemma 3.7. These results will be extremely useful in the next section.

Lemma 3.8. *Let $\lambda_{k_0}\in\sigma(-\Delta;\Omega)$ and $\alpha_{j_0}\in\sigma(A)\setminus\{0\}$. Then the following conditions are equivalent*

(1) $\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)=\Theta\in U(SO(2))$,
(2) $V_{-\Delta}(\lambda_{k_0})$ *is a trivial $SO(2)$–representation and $\mu(\alpha_{j_0})\cdot\mu(\lambda_{k_0})$ is even.*

Proof. (1) $\Rightarrow$ (2) Taking into account Lemmas 2.2, 3.7 and (2.2) we obtain the following

$$\left[\prod_{\alpha_j\in\sigma_+(A)}\left(\nabla_{SO(2)}-\deg\left(-Id,D_\beta\left(\mathcal{W}\left(\alpha_j,\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)\right)\right)\right)^{\mu(\alpha_j)}\right]_{SO(2)}=\pm 1. \tag{3.5}$$

Since $\left(\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)\right)_{SO(2)}=0$ and (2.2), (3.5),

$$\left(\left(\nabla_{SO(2)}-\deg(-Id,D_\beta(V_{-\Delta}(\lambda_{k_0})))\right)^{\mu(\alpha_{j_0})}-\mathbb{I}\right)_{SO(2)}=0 \tag{3.6}$$

Taking into consideration Lemma 2.2 and (2.2) we obtain that equality (3.6) holds true iff $\mu(\alpha_{j_0})\cdot\mu(\lambda_{k_0})$ is an even number. Moreover, since (2.2), (3.5) and (3.6) we have

$$\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)=\pm\left[\left(\nabla_{SO(2)}-\deg(-Id,D_\beta(V_{-\Delta}(\lambda_{k_0})))\right)^{\mu(\alpha_{j_0})}-\mathbb{I}\right] \tag{3.7}$$

Combining $\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)=\Theta$ with (3.7) we obtain that $V_{-\Delta}(\lambda_{k_0})$ is a trivial representation of the group $SO(2)$.

(2) $\Rightarrow$ (1) Since $\mu(\alpha_{j_0})\cdot\mu(\lambda_{k_0})$ is even, equality (3.6) holds true. Combining (3.5) with (3.6) we obtain (3.7). Since $V_{-\Delta}(\lambda_{k_0})$ is a trivial $SO(2)-$representation and (3.7),

$$\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)=\Theta\in U(SO(2)),$$

which completes the proof. $\square$

Notice that we can formulate Lemma 3.8 in the following equivalent way.

Remark 3.2. *Let $\lambda_{k_0}\in\sigma(-\Delta;\Omega)$ and $\alpha_{j_0}\in\sigma(A)\setminus\{0\}$. Then $\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)\neq\Theta\in U(SO(2))$ iff one of the following conditions is fulfilled*

1. *$\mu(\lambda_{k_0})\cdot\mu(\alpha_{j_0})$ is odd,*
2. *$V_{-\Delta}(\lambda_{k_0})$ is a nontrivial representation of the group $SO(2)$.*

Remark 3.2 justify the following definition.

Definition 3.1. *A pair $(\lambda_{k_0},\alpha_{j_0})\in\sigma(-\Delta;\Omega)\times(\sigma(A)\setminus\{0\})$ is said to be essential if $V_{-\Delta}(\lambda_{k_0})$ is a nontrivial representation of the group $SO(2)$ or $\mu(\lambda_{k_0})\cdot\mu(\alpha_{j_0})$ is odd.*

In Lemmas 3.9, 3.10 we formulate sufficient conditions for which the bifurcation index is an element of $U_\pm(SO(2))$.

Lemma 3.9. *Fix $\lambda_{k_0}\in\sigma(-\Delta;\Omega)$ and $\alpha_{j_0}\in\sigma_\pm(A)$. Moreover, assume that*

$$\mu(\alpha_{j_0})\cdot\mu(\lambda_{k_0})+\sum_{\alpha\in\sigma_\pm(A)}\dim\mathcal{W}\left(\alpha,\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)\cdot\mu(\alpha)$$

is even. Then $\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)\in U_\pm(SO(2))$.

Proof. Assume that $\alpha_{j_0} \in \sigma_+(A)$. Suppose that both numbers

$$\mu(\alpha_{j_0}) \cdot \mu(\lambda_{k_0}), \sum_{\alpha \in \sigma_+(A)} \dim \mathcal{W}\left(\alpha, \frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \cdot \mu(\alpha)$$

are even. By Lemmas 2.1 , 2.2 and (2.2) we obtain the following

$$\left(\nabla_{SO(2)} - \deg(-Id, D_\beta(V_{-\Delta}(\lambda_{k_0})))\right)^{\mu(\alpha_{j_0})} - \mathbb{I} \in U_+(SO(2)) \tag{3.8}$$

$$\prod_{\alpha \in \sigma_+(A)} \left(\nabla_{SO(2)} - \deg\left(-Id, D_\beta\left(\mathcal{W}\left(\alpha, \frac{\lambda_{k_0}}{\alpha_{j_0}}\right)\right)\right)\right)^{\mu(\alpha)} \in U_+(SO(2)) \tag{3.9}$$

Combining (3.8), (3.9) with Lemma 3.7 and Remark 2.1 we obtain that $\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \in U_+(SO(2))$.

Suppose that both numbers $\mu(\alpha_{j_0}) \cdot \mu(\lambda_{k_0}), \sum_{\alpha \in \sigma_+(A)} \dim \mathcal{W}\left(\alpha, \frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \cdot \mu(\alpha)$ are odd. By Lemmas 2.1, 2.2 and (2.2) we obtain the following

$$\left(\nabla_{SO(2)} - \deg(-Id, D_\beta(V_{-\Delta}(\lambda_{k_0})))\right)^{\mu(\alpha_{j_0})} - \mathbb{I} \in U_-(SO(2)) \tag{3.10}$$

$$\prod_{\alpha \in \sigma_+(A)} \left(\nabla_{SO(2)} - \deg\left(-Id, D_\beta\left(\mathcal{W}\left(\alpha, \frac{\lambda_{k_0}}{\alpha_{j_0}}\right)\right)\right)\right)^{\mu(\alpha)} \in U_-(SO(2)) \tag{3.11}$$

Combining (3.10), (3.11) with Lemma 3.7 and Remark 2.1 we obtain that $\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \in U_+(SO(2))$.

Assume that $\alpha_{j_0} \in \sigma_-(A)$. Repeating in this case in fact the same proof as above we obtain $\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \in U_-(SO(2))$. □

Proof of Lemma 3.10 is very similar to the proof of Lemma 3.9. Therefore it is left to the reader.

Lemma 3.10. *Fix $\lambda_{k_0} \in \sigma(-\Delta; \Omega)$ and $\alpha_{j_0} \in \sigma_\pm(A)$. Moreover, assume that*

$$\mu(\alpha_{j_0}) \cdot \mu(\lambda_{k_0}) + \sum_{\alpha \in \sigma_\pm(A)} \dim \mathcal{W}\left(\alpha, \frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \cdot \mu(\alpha)$$

is odd. Then $\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \in U_\mp(SO(2))$.

4. Nonlinear Equation

In this section we study weak solutions of the system of nonlinear elliptic differential equations of the form

$$\begin{cases} -\Delta u &= \nabla_u F(u, \lambda) \quad \text{in } \Omega \\ u &= (u_1, \ldots, u_m) = 0 \quad \text{on } \partial\Omega \end{cases} \tag{4.1}$$

where

(1) $\Omega \subset \mathbb{R}^N$ is an open, bounded, $SO(2)-$invariant subset of an orthogonal $SO(2)-$representation $\mathbb{R}^N$, with boundary of the class C^{1-},
(2) $F \in C^2(\mathbb{R}^m \times \mathbb{R}, \mathbb{R})$,
(3) $F(x, \lambda) = \dfrac{\lambda}{2}\langle Ax, x\rangle + \eta(x, \lambda)$, where
  (a) A is a symmetric $(m \times m)-$matrix,
  (b) $\nabla_x \eta(0, \lambda) = 0$, for any $\lambda \in \mathbb{R}$,
  (c) $\nabla_x^2 \eta(0, \lambda) = 0$, for any $\lambda \in \mathbb{R}$,
(4) for any $\lambda \in \mathbb{R}$ there are $C_\lambda > 0$ and $1 \leq p_\lambda < (N+2)(N-2)^{-1}$ such that for any $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}$ the following inequality holds true $\mid \nabla_x F(x, \lambda) \mid \leq C_\lambda\, (1+ \mid x \mid^{p_\lambda})$.

Consider Hilbert space $\mathbb{H} = \displaystyle\bigoplus_{i=1}^{m} \mathbb{H}_0^1(\Omega)$ with scalar product

$$\langle u_1, u_2\rangle_{\mathbb{H}} = \sum_{i=1}^{m} \langle u_{1,i}, u_{2,i}\rangle_{\mathbb{H}_0^1(\Omega)} = \sum_{i=1}^{m} \int_\Omega (\nabla u_{1,i}(x), \nabla u_{2,i}(x))\; dx,$$

where $u_i = (u_{i,1}, u_{i,2}, \dots, u_{i,m}) \in \mathbb{H}, i = 1, 2$ and $(\cdot, \cdot)$ is the usual scalar product in $\mathbb{R}^m$.

Remark 4.1. *It is known that* $(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ *is an orthogonal* $SO(2)-$*representation with* $SO(2)-$*action given by* $g(u(x)) = u(gx)$.

Define functional $\Phi : \mathbb{H} \times \mathbb{R} \to \mathbb{R}$ as follows $\Phi(u, \lambda) = \dfrac{1}{2}\langle u, u\rangle_{\mathbb{H}} - \displaystyle\int_\Omega F(u, \lambda)\, dx.$

Remark 4.2. *Under the above assumptions we have* $\Phi \in C^2_{SO(2)}(\mathbb{H} \times \mathbb{R}, \mathbb{R})$. *Moreover, critical points of* Φ *(with respect to* u*) are in one-to-one correspondence with weak solutions of system* (4.1).

Notice that for $u = (u_1, \dots, u_m), \varphi = (\varphi_1, \dots, \varphi_m) \in \mathbb{H}$ we have

$$\langle \nabla_u \Phi(u, \lambda), \varphi\rangle_{\mathbb{H}} = D_u\Phi(u, \lambda)(\varphi) = \langle u - \lambda L_A(u) - K(u, \lambda), \varphi\rangle_{\mathbb{H}}.$$

where

(1) $L_A = A \cdot ((-\Delta)^{-1} \cdot Id_{\mathbb{R}^m}) : \mathbb{H} \to \mathbb{H}$ is linear, bounded, compact, self-adjoint and $SO(2)-$equivariant operator given by $\langle L_A u, v\rangle_{\mathbb{H}} = \displaystyle\int_\Omega Au(x)v(x)dx.$
(2) $K : \mathbb{H} \times \mathbb{R} \to \mathbb{H}$ is a compact, $SO(2)-$equivariant operator such that $K(0, \lambda) = 0$ and $DK(0, \lambda) = 0$ for any $\lambda \in \mathbb{R}$.

The following theorem is the Rabinowitz global bifurcation theorem for systems of elliptic differential equations considered on $SO(2)-$invariant domains. It yields information about global behaviour of connected sets of weak solutions of system (4.1). This theorem deliver sufficient conditions for the existence of branching points of weak solutions of system (4.1).

Theorem 4.1. *Fix an essential pair* $(\lambda_{k_0}, \alpha_{j_0}) \in \sigma(-\Delta; \Omega) \times \sigma(A)$. *Then*

(1) *either* $\mathcal{C}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)$ *is unbounded in* $\mathbb{H} \times \mathbb{R}$,

(2) *or* $\mathcal{C}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)$ *is bounded in* $\mathbb{H}\times\mathbb{R}$. *Moreover, the following conditions are satisfied*

(a) $\mathcal{C}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)\cap(\{0\}\times\mathbb{R})=\{0\}\times\left(\bigcup_{s=0}^{p}\left\{\frac{\lambda_{k_s}}{\alpha_{j_s}}\right\}\right)$, *where*

(i) $\lambda_{k_0},\lambda_{k_1},\ldots,\lambda_{k_p},\in\sigma(-\Delta;\Omega)$,

(ii) $\alpha_{j_0},\alpha_{j_1},\ldots,\alpha_{j_p}\in\sigma(A)\setminus\{0\}$,

(b)

$$\sum_{s=0}^{p}\mathcal{BIF}\left(\frac{\lambda_{k_s}}{\alpha_{j_s}}\right)=\Theta\in U(SO(2)) \tag{4.2}$$

Proof. To prove this theorem it is enough to study zeros of the operator $\nabla_u\Phi:\mathbb{H}\times\mathbb{R}\to\mathbb{H}$. From Lemma 3.2 it follows that $\nabla_u^2\Phi\left(0,\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)=Id-\frac{\lambda_{k_0}}{\alpha_{j_0}}L_A$ is not an isomorphism. By assumptions and Lemma 3.2 we obtain that $\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)\neq\Theta$. The rest of the proof is a direct consequence of Theorem 2.5. $\square$

Below we prove two consequences of Theorem 4.1. Namely, in Corollary 4.1 we formulate conditions which exclude one of the possibilities in the alternative given by Theorem 4.1 can be excluded. Moreover, in Corollary 4.2 we describe some properties of bounded continua.

Corollary 4.1. *Let pair* $(\lambda_{k_0},\alpha_{j_0})\in\sigma(-\Delta;\Omega)\times\sigma(A)$ *be essential. Assume additionally that*

(1) $\sigma_{\mp}(A)=\emptyset$,

(2) $\mu(\alpha_j)$ *is even for any* $\alpha_j\in\sigma_{\pm}(A)$.

Then continuum $\mathcal{C}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)$ *is unbounded in* $\mathbb{H}\times\mathbb{R}$.

Proof. Suppose that $\sigma_-(A)=\emptyset$. By Lemma 3.9 it follows that $\mathcal{BIF}\left(\frac{\lambda_k}{\alpha_j}\right)\in U_+(SO(2))$ for any $\lambda_k\in\sigma(-\Delta;\Omega)$ and $\alpha_j\in\sigma_+(A)$. Since pair $(\lambda_{k_0},\alpha_{j_0})\in\sigma(-\Delta;\Omega)\times\sigma_+(A)$ is essential, $\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)\neq\Theta\in U(SO(2))$. Summing up, condition (4.2) in Theorem 4.1 can never be satisfied. Suppose now that $\sigma_+(A)=\emptyset$. The proof in this case is in fact the same as the proof of the case $\sigma_-(A)=\emptyset$. $\square$

In the following corollary we describe bounded continua of nontrivial solutions of (4.1).

Corollary 4.2. *Let* $(\lambda_{k_0},\alpha_{j_0})\in\sigma(-\Delta;\Omega)\times\sigma(A)$ *be an essential pair. Assume additionally that*

(1) $\alpha_{j_0}\in\sigma_{\pm}(A)$,

(2) $\mu(\alpha_j)$ *is even for any* $\alpha_j\in\sigma_{\pm}(A)$.

If continuum $\mathcal{C}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)$ *is bounded in* $\mathbb{H}\times\mathbb{R}$ *then*

$$\sigma_{\mp}(A)\neq\emptyset \quad\text{and}\quad \mathcal{C}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)\cap\left(\{0\}\times\left(\bigcup_{\alpha_j\in\sigma_{\mp}(A)}\ \bigcup_{\lambda_k\in\sigma(-\Delta;\Omega)}\left\{\frac{\lambda_k}{\alpha_j}\right\}\right)\right)\neq\emptyset.$$

Proof. Suppose that $\alpha_{j_0} \in \sigma_+(A)$. By assumption and Lemma 3.9 it follows that

(1) $\frac{\lambda_{k_0}}{\alpha_{j_0}} > 0$,
(2) $\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \neq \Theta \in U(SO(2))$,
(3) $\mathcal{BIF}\left(\frac{\lambda_k}{\alpha_j}\right) \in U_+(SO(2))$ any $\lambda_k \in \sigma(-\Delta;\Omega)$ and $\alpha_j \in \sigma_+(A)$.

Suppose the assertion of the lemma is false. Hence

$$\sigma_-(A) = \emptyset \text{ or } \mathcal{C}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \cap (\{0\}\times\mathbb{R}) \subset \{0\}\times(0,+\infty) \tag{4.3}$$

Taking into account $(1)-(3)$ and (4.3) we show that equality equality (4.2) in Theorem 4.1 can never be satisfied, a contradiction. In fact we can repeat the above proof for $\alpha_{j_0} \in \sigma_-(A)$. □

5. Illustration

In this section we illustrate the abstract results proved in the previous section. Let $B^n \subset \mathbb{R}^n$ denote an open disc of radius 1 centered at the origin. We will study system (4.1) with $\Omega = B^2$ or $\Omega = B^3$.

First of all let us describe eigenspaces of $-\Delta$ as representations of the group $SO(2)$.

Put $\Omega = B^2$. It is known that if $\lambda_{k_0} \in \sigma(-\Delta; B^2)$ then there exists $k \in \mathbb{N}\cup\{0\}$ such that $V_{-\Delta}(\lambda_{k_0}) \approx \mathbb{R}[1,k]$. In other words any eigenspace of $-\Delta$ is either one-dimensional trivial or two-dimensional nontrivial representation of the group $SO(2)$.

Put $\Omega = B^3$. We know that if $\lambda_{k_0} \in \sigma(-\Delta; B^3)$, then there exists $k \in \mathbb{N}\cup\{0\}$ such that $V_{-\Delta}(\lambda_{k_0}) \approx \mathbb{R}[1,0]\oplus\mathbb{R}[1,1]\oplus ... \oplus\mathbb{R}[1,k]$.

From now on let $\Omega = B^2$ or $\Omega = B^3$. Therefore for fixed $\lambda_{k_0} \in \sigma(-\Delta;\Omega)$ we have the following assertion

(i) if $\mu(\lambda_{k_0}) > 1$, then for any $\alpha_{j_0} \in \sigma(A)\setminus\{0\}$ the pair $(\lambda_{k_0}, \alpha_{j_0})$ is essential,
(ii) if $\mu(\lambda_{k_0}) = 1$, then the pair $(\lambda_{k_0}, \alpha_{j_0})$ is essential iff $\mu(\alpha_{j_0})$ is odd.

Remark 5.1. *Let us fix a pair $(\lambda_{k_0}, \alpha_{j_0}) \in \sigma(-\Delta;\Omega)\times(\sigma(A)\setminus\{0\})$ such that $\mu(\lambda_{k_0}) > 1$ or $\mu(\alpha_{j_0})$ is odd. Then all the assumptions of Theorem 4.1 are fulfilled. Therefore $\left(0, \frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \in \mathbb{H}\times\mathbb{R}$ is a branching point of weak solutions of system* (4.1). *Moreover, continuum $\mathcal{C}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \subset \mathbb{H}\times\mathbb{R}$ satisfies thesis of Theorem 4.1.*

As a direct consequence of Corollary 4.1 we obtain the following remark.

Remark 5.2. *Assume that $\sigma_\mp(A) = \emptyset$ and that $\mu(\alpha)$ is even for any $\alpha \in \sigma_\pm(A)$. If $(\lambda_{k_0}, \alpha_{j_0}) \in \sigma(-\Delta;\Omega)\times\sigma_\pm(A)$ is a pair such that $\mu(\lambda_{k_0}) > 1$, then continuum $C\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)$ is unbounded in $\mathbb{H}\times\mathbb{R}$.*

The following remark is a consequence of Corollary 4.2.

Remark 5.3. *Assume that* $\mu(\alpha)$ *is even for any* $\alpha \in \sigma_\pm(A)$. *Let us fix* $(\lambda_{k_0}, \alpha_{j_0}) \in \sigma(-\Delta; \Omega) \times \sigma_\pm(A)$ *such that* $\mu(\lambda_{k_0}) > 1$. *If continuum* $C\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right)$ *is bounded in* $\mathbb{H} \times \mathbb{R}$ *then*

$$\sigma_\mp(A) \neq \emptyset \text{ and } C\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \cap \left(\{0\} \times \left(\bigcup_{\alpha_j \in \sigma_\mp(A)} \bigcup_{\lambda_k \in \sigma(-\Delta;\Omega)} \left\{\frac{\lambda_k}{\alpha_j}\right\}\right)\right) \neq \emptyset.$$

6. Final Remarks

Fix an inessential pair $(\lambda_{k_0}, \alpha_{j_0}) \in \sigma(-\Delta; \Omega) \times (\sigma(A) \setminus \{0\})$. In this situation we can not apply Theorem 4.1 because $\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) = \Theta \in U(SO(2))$. On the other hand using the Lapunov-Schmidt reduction we can locally convert the problem of finding of bifurcation points of solutions of system (4.1) to a finite-dimensional one. Next using the finite-dimensional Morse theory (or the Conley index technique) we can prove that $\left(0, \frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \in \mathbb{H} \times \mathbb{R}$ is a bifurcation point of solutions of system (4.1). It is worth to point out that it can happen that this point is not a branching point of solutions of system (4.1), see [2, 3, 15, 19, 32] for examples and discussion.

Notice that to study weak solutions of system (4.1) one can also apply the classical Rabinowitz global bifurcation theorem, see for instance [16, 20, 23, 24]. In other words one can forget about $SO(2)-$invariance of Ω and variational structure of system (4.1). To apply the Rabinowitz alternative we have to compute the bifurcation index $\mathcal{BIF}_{LS}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \in \mathbb{Z}$ in terms of the Leray-Schauder degree. However, for a pair $(\lambda_{k_0}, \alpha_{j_0}) \in \sigma(-\Delta; \Omega) \times (\sigma(A) \setminus \{0\})$ such that $\mu(\alpha_{j_0}) \cdot \mu(\lambda_{k_0})$ is even we have $\mathcal{BIF}_{LS}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) = 0 \in \mathbb{Z}$. If in addition $V_{-\Delta}(\lambda_{k_0})$ is a nontrivial $SO(2)-$representation then $\mathcal{BIF}\left(\frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \neq \Theta \in U(SO(2))$. In other words if a pair $(\lambda_{k_0}, \alpha_{j_0}) \in \sigma(-\Delta; \Omega) \times (\sigma(A) \setminus \{0\})$ is essential and such that $\mu(\alpha_{j_0}) \cdot \mu(\lambda_{k_0})$ is even then assumptions of Theorem 4.1 are fulfilled and the classical Rabinowitz alternative is not applicable. Moreover, for the same reason the classical Rabinowitz alternative is not applicable under the assumptions of Corollary 4.1 at any point $\left(0, \frac{\lambda_{k_0}}{\alpha_{j_0}}\right) \in \mathbb{H} \times \mathbb{R}$, where $(\lambda_{k_0}, \alpha_{j_0}) \in \sigma(-\Delta; \Omega) \times (\sigma(A) \setminus \{0\})$.

We show in Corollary 4.1 that under some easy to verify assumptions there are unbounded continua of nontrivial solutions of system (4.1). It is clear that Corollary 4.1 is a generalization of Theorem 3.3 of [29]. Namely, if the number m of equations in system (4.1) is even and $A = Id_{\mathbb{R}^m}$ then we obtain Theorem 3.3 of [29].

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Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, PL-87-100 Toruń, ul. Chopina 12/18, Poland
E-mail address: tosca@mat.uni.torun.pl

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, PL-87-100 Toruń, ul. Chopina 12/18, Poland
E-mail address: Slawomir.Rybicki@mat.uni.torun.pl